

Contents

Preface	v
1 Triangulations in Mathematics	1
1.1 Combinatorics and triangulations	2
1.2 Optimization and triangulations	13
1.3 Algebra and triangulations	21
1.4 The rest of this book	34
<i>Exercises</i>	38
2 Configurations, Triangulations, Subdivisions, and Flips	43
2.1 The official languages in the land of triangulations	43
2.1.1 Polyhedra and cones	43
2.1.2 Point configurations	47
2.1.3 Geometry of point configurations	50
2.2 A closer look at the definition of triangulation	53
2.2.1 There is always a triangulation	54
2.2.2 A famous example: the Delaunay triangulation	56
2.2.3 Regular subdivisions and their structure	59
2.3 A bullet-proof definition of polyhedral subdivisions	62
2.3.1 Polyhedral subdivisions	62
2.3.2 Regular subdivisions, again	67
2.4 Flips and the graph of triangulations	72
2.4.1 Corank-one configurations and circuits	72
2.4.2 Almost-triangulations and flips	74
2.5 Vector configurations and their triangulations	76
2.5.1 Vector configurations	77
2.5.2 Polyhedral subdivisions of vector configurations	79
2.5.3 Regular subdivisions of vector configurations	81
2.6 Triangulations as simplicial complexes	83
2.6.1 Simplicial complexes	83
2.6.2 The f -vector of a simplicial complexes	84
2.6.3 Linear constraints on the f -vector	87
<i>Exercises</i>	90
3 Life in Two Dimensions	93
3.1 Some basic properties	93
3.2 A few examples of triangulations in the plane	95
3.2.1 Placing and pulling triangulations	96
3.2.2 Delaunay triangulations	97
3.2.3 Greedy and minimum weight triangulations	102
3.3 The set of all triangulations of a point set	107
3.3.1 The exact number of triangulations	107

3.3.2	The maximum possible number of triangulations	112
3.3.3	The minimum possible number of triangulations	115
3.3.4	The poset of subdivisions	116
3.4	Flips in triangulations	119
3.4.1	All triangulations of a point set in the plane are connected by flips	120
3.4.2	Effective enumeration of triangulations	123
3.4.3	Further properties of the graph of flips	128
3.5	Pseudo-triangulations	131
3.6	Life in three dimensions	133
3.6.1	The number of tetrahedra	134
3.6.2	Monotone flipping does not (always) work	137
3.6.3	The number of flips	141
3.7	Notes and References	145
	<i>Exercises</i>	146
4	A Tool Box	149
4.1	Combinatorics of configurations	149
4.1.1	Dependences, circuits, and the intersection property	150
4.1.2	Evaluations, cocircuits, and the union property	155
4.1.3	Gale transforms and the duality between circuits and cocircuits	160
4.2	Manipulating vector configurations	165
4.2.1	Pyramids and joins	165
4.2.2	Prisms and products	167
4.2.3	Deletion	169
4.2.4	Contraction	171
4.2.5	One-point suspension	175
4.3	Generating polyhedral subdivisions	178
4.3.1	The placing (or pushing) triangulation	178
4.3.2	The pulling triangulation	181
4.3.3	Lexicographic triangulations	182
4.3.4	Pushing and pulling refinements	183
4.4	Two equivalent characterizations of flips	185
4.4.1	Flips via circuits	186
4.4.2	Flips via walls	188
4.5	More characterizations of triangulations and subdivisions	190
4.5.1	Geometric characterizations	191
4.5.2	Combinatorial characterizations	203
	<i>Exercises</i>	207
5	Regular Triangulations and Secondary Polytopes	209
5.1	The secondary polytope	210
5.1.1	Motivating examples	210
5.1.2	Statement of the main theorem	214
5.1.3	Dimension and affine span of the secondary polytope	217
5.2	The normal fan of the secondary polytope	221
5.2.1	Secondary cones	221
5.2.2	The secondary fan	225
5.2.3	Proof of the main theorem	229

5.3	Structure of the secondary polytope	233
5.3.1	Edges of the secondary polytope	233
5.3.2	Monotone paths on the secondary polytope	236
5.3.3	Facets of the secondary polytope	241
5.4	Chambers	243
5.4.1	The chamber fan	243
5.4.2	Flips in the chamber fan	248
5.5	Configurations with fixed corank	257
5.5.1	Configurations with $d + 3$ points	257
5.5.2	Configurations with $d + 4$ points	261
5.5.3	Lawrence polytopes and the complexity of secondary polytopes	264
	<i>Exercises</i>	270
6	Some Interesting Configurations	275
6.1	Cyclic polytopes	275
6.1.1	Warm-up example: two dimensions	276
6.1.2	Combinatorial properties of cyclic polytopes	277
6.1.3	Triangulations as sections of the canonical projection	283
6.1.4	Higher Stasheff-Tamari posets	285
6.1.5	The structure theorem for the first Stasheff-Tamari poset	287
6.1.6	Cyclic polytopes have many triangulations	290
6.2	Products of two simplices	294
6.2.1	The prism over a simplex	294
6.2.2	The product of simplices	299
6.2.3	Staircase triangulations	301
6.2.4	Non-regular triangulations of products of simplices	304
6.3	Cubes and their subpolytopes	311
6.3.1	Small 0/1 non-regular triangulations	311
6.3.2	Two simple ways to triangulate any cube	314
6.3.3	Triangulating high-dimensional cubes. State of the art	316
6.3.4	Cubes of three dimensions	319
6.3.5	Cubes of four dimensions	322
6.3.6	Slices of cubes: triangulations of hypersimplices	325
6.3.7	Birkhoff's polytope	330
	<i>Exercises</i>	334
7	Some Interesting Triangulations	337
7.1	The mother of all examples, and some relatives	338
7.1.1	A theme with many variations	338
7.1.2	Twelve proofs of non-regularity	342
7.2	Highly flip-deficient triangulations	345
7.2.1	Dimension 3: A zig-zag grid	345
7.2.2	Locally acyclic orientations and triangulations of products	349
7.2.3	Locally acyclic orientations without reversible edges	352
7.2.4	Dimension 4: Layers of prisms	356
7.3	Dimension 5: A disconnected graph of triangulations with unimodular triangulations	358
7.3.1	Locally acyclic orientations of boundary subcomplexes	358
7.3.2	Unimodular triangulations in different components of the graph of triangulations	360

7.3.3	Exponential number of components in the graph of flips	361
7.4	Dimension 6: A disconnected graph of triangulations in general position	362
7.4.1	The building block: Gale octagons	363
7.4.2	Seventeen points in special position	365
7.4.3	A disconnected space of triangulations in general position	369
	<i>Exercises</i>	374
8	Algorithmic Issues	377
8.1	Tools for computation	377
8.1.1	Chirotopes	377
8.1.2	Computing the chirotope	378
8.1.3	Computing circuit and cocircuit signatures from the chirotope	383
8.2	Verification and realizability	385
8.2.1	Constructing regular triangulations in practice	386
8.2.2	Checking regularity of a triangulation	387
8.3	Listing and enumerating triangulations	388
8.3.1	Exploring a flip-graph component	389
8.3.2	Enumeration of all triangulations	390
8.3.3	Enumeration with symmetry	392
8.3.4	Implementation issues	393
8.4	Bounding the number of triangulations	396
8.5	Optimization	398
8.5.1	A linear optimization approach: the universal polytope	400
8.5.2	Relaxations of the universal polytope and its edges	406
8.5.3	Equidecomposable and weakly neighborly polytopes	410
8.6	Computational complexity of triangulation problems	413
8.6.1	A very quick review of complexity classes	413
8.6.2	The hardness of the planar constrained triangulation problem	415
8.6.3	Hardness of minimum length triangulations in the plane	422
8.6.4	Hardness of minimal size triangulations of convex polytopes	425
	<i>Exercises</i>	429
9	Further Topics	433
9.1	Fiber polytopes	433
9.1.1	Monotone paths	433
9.1.2	Zonotopal tilings	436
9.1.3	Polyhedral subdivisions	438
9.1.4	Compatible subdivisions and the fiber polytope	438
9.2	Mixed subdivisions and the Cayley trick	445
9.2.1	An example	445
9.2.2	Mixed subdivisions and the Minkowski projection	447
9.2.3	Subdivisions in the Cayley embedding and the Cayley projection	452
9.2.4	The Cayley trick	454
9.2.5	Product of a triangle and k -simplex	459
9.3	Lattice polytopes and unimodular triangulations	463
9.3.1	Triangulations of lattice polygons	465
9.3.2	Existence of unimodular triangulations	469
9.3.3	Ehrhart polynomials and unimodular triangulations	475

9.4	Triangulations and Gröbner bases	478
9.4.1	Gröbner bases and toric ideals	479
9.4.2	Sturmfels' correspondence	481
9.5	Polytopal complexes and regular triangulations	488
9.5.1	Central and normal fans as regular triangulations	489
9.5.2	Shellings, flips, and face vectors	493
9.5.3	Polytopality via regular triangulations	502
	<i>Exercises</i>	509
Bibliography		513
Index		531